Math 10B with Professor Stankova
Worksheet, Discussion #28; Thursday, 5/3/2018
GSI name: Roy Zhao

1 Permutations/Combinations/12-Fold Way

1.1 Counting

1. How many ways are there to rearrange the letters of $ZVEZDA$?

2. What is the coefficient of $x^2y^3z^5$ in $(x/2+3y-2z)^{10}$? What about the coefficient of $x^3y^3z^3$?

3. How many ways can I create a license plate that has 3 letters followed by 3 numbers if I want exactly 1 I and at least 1 1.

4. How many 5 card hands out of a standard 52 card deck have 4 different suits?

5. I am baking cookies for Alice Bob and Carol. Each want at least 1 cookie but Bob wants at least 3 cookies. Alice is on a diet and wants at most 3 cookies. How many ways can I divide the 10 cookies I made amongst them?

6. In the previous cookie problem, let X be the most number of cookies any one of Alice, Bob, or Carol gets (if they got $2, 3, 5$ cookies, then $X = 5$), what can we say about the minimum X can be?

1.2 Probability/Expected Value

7. Let X be a random variable on a probability space Ω with a probability function P and let f be the PMF for X. Draw a picture of how all these variables interact and explain any special arrows that you have in your diagram.

8. When I roll a fair 6 sided die 10 times, what is the expected number of distinct numbers that appear? (For instance, if I roll $1, 1, 3, 3, 2$, there are 3 distinct numbers that appear)

9. Eve has 5 cards in her hand and I know that one of them is the ace of spades. What is the probability that she has a pair of aces (exactly 2 aces)?

10. A red-green colorblind person picks an apple out of a bag. There are 4 red apples and 1 green apple. With probability 3/4 he says the correct color of the apple he picked out. What is the probability that he says that the apple he picks out is red?

11. In the previous apple problem, what is the probability that the apple is actually red when he says it is red?

2 Distributions

12. Suppose that I have a weighted die that lands on $1, 2, 3, 4, 5$ with equal probability and 6 5 times as likely as 1. Let X be the value of the die. What is the PMF for X?

13. For my weighted die in the previous problem, what is the probability in 10 rolls, I roll a 5 or 6 exactly 6 times? What kind of distribution is this?

14. For my weighted die in the previous problem, suppose that I keep rolling until I roll a 5 or 6. What is the expected number of times I need to roll the die? What kind of distribution is this?

15. Suppose that X is binomially distributed with $E[X] = 15$ and $Var(X) = 6$. How many trials n are there and what is the probability p of success?

16. Suppose that the number of students who fill out course evaluations per day is Poisson distributed and on average 2 students fill out evaluations per day. What is the probability that in a week, no students fill out evaluations? What is the probability that in a week, 7 people fill out evaluations?

3 Hypothesis Testing/CLT

17. I have a (possibly biased) coin and flip it 100 times and get heads 90 times. What is the 95% confidence interval for p, the probability of flipping a heads?

18. When counting families with 2 children, I find that 83 of them have two girls, 102 of them have two boys, and 215 of them have one boy and one girl. Suppose that my null hypothesis is having a boy or girl is equally likely and the two children's genders are independent of each other. What kind of test should I use to test this hypothesis? Perform this test and explain what kind of table to look up as well as what value to look at.

19. Every day, the number of people who are born is Poisson distributed with an average of 4900 people per day. We count how many people are born in a span of 100 days and let $\bar{X}$ denote the average number of people born per day. What is the probability $P(\bar{X} \leq 4895)$?

20. What is the definition of variance? Prove that $Var(X) = E[X^2] - E[X]^2$.

21. What is the definition of covariance? Prove that $Cov(X, Y) = E[XY] - E[X]E[Y]$.

4 First Order Differential Equations

22. True False A differential equation $y' = f(y, t)$ with an initial condition $y(0) = y_0$ will always have a unique solution.

23. Solve the differential equations $(t^3 + t^2)y' = \frac{t^2+2t+2}{2y}$ with the initial condition $y(1) = 1$.

24. Consider the differential equation $ty' + 3y = 5t^2$ with initial condition $y(1) = 1$. Draw a slope field and then estimate $y(5)$ using a step size of $h = 2$. Then solve for y explicitly and find the exact value of $y(5)$.

25. Find all solutions to $e^t y' = y^2 + 2y + 1$.

26. Newton's law of cooling says that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature. Suppose that I put a frozen pizza initially at $0°C$ into an oven with a temperature of $200°C$ and after half an hour the pizza has reached a temperature of $100°$. Find the temperature $T(t)$ of the pizza after t minutes.

5 Recurrence Relations and 2nd order Differential Equations

27. True False It is possible for an IVP to have a unique solution.

28. True False It is possible for a BVP to have a unique solution.

29. True False It is possible for an IVP to have infinitely many solutions.

30. True False It is possible for a BVP to have infinitely many solutions.

31. Solve the recursion equation $a_n = 2a_{n-2} - a_{n-1}$ with the initial conditions $a_0 = 0, a_1 = 3$.

32. Verify that $y_1(t) = t$ and $y_2(t) = t^3$ are solutions to the differential equation $t^2y''(t) - 3ty'(t) + 3y(t) = 0$. Find the solution to the differential equation with $y(1) = 2$ and $y'(1) = 4$ (hint: what kind of differential equation is this?).

33. Find all solutions to the BVP $y'' + 2y' + 5y = 0$ with $y(0) = 0$ and $y(\pi) = 0$.

34. Find all solutions to the BVP $y'' - 5y' + 6y = 0$ with $y(0) = 2$ and $y(1) = e^2 + e^3$.

35. Find a second order differential equation IVP that has te^t as a solution.

36. Find a second order differential equation BVP that has $e^{2t}\sin(t)$ as a solution.

6 Matrices and Correlation

37. True False If A, B are square $n \times n$ matrices, then $AB = BA$.

38. True False If A is a 2×2 matrix such that $A^2 = I_2$, then $A = I_2$.

39. Find the solution to
$$\begin{cases} y_1'(t) = -y_1(t) - 5y_2(t) \\ y_2'(t) = 2y_1(t) + y_2(t) \end{cases}$$
with $y_1(0) = -5$ and $y_2(0) = -2$.

40. Consider the following set of points: $\{(0,6),(1,3),(2,1),(3,0),(4,0)\}$. Find the line of best fit through these points and use it to estimate $y(0.5)$. What is the correlation of the data?

41. Let $A = \begin{pmatrix} 3 & 4 & -1 \\ 4 & 2 & 1 \\ -2 & -3 & 1 \end{pmatrix}$. Find A^{-1}.

42. Let A be the same as the previous problem. Solve $A\vec{x} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ (hint: use the previous problem to do this quickly).

43. Let $\vec{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ and suppose that A is a 3×3 matrix such that $A\vec{v}_1 = 4\vec{v}_1, A\vec{v}_2 = \vec{0}, A\vec{v}_3 = -\vec{v}_3$. What are the eigenvalues and eigenvectors of A? What is the general solution to $\vec{y}'(t) = A\vec{y}$ with $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$?

44. Let A be a 2×2 matrix and suppose that $\vec{y} = \begin{pmatrix} 3e^{2t} + 4e^{4t} \\ e^{4t} - e^{2t} \end{pmatrix}$ is a solution to $\vec{y}' = A\vec{y}$. What are the eigenvalues and eigenvectors of A? What is $A\begin{pmatrix} 3 \\ -1 \end{pmatrix}, A\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 7 \\ 0 \end{pmatrix}$?

45. Find the line of best fit through the points $\{(0,2),(1,3),(2,1)\}$ and the correlation of the data.

46. Write the differential equation $y'' + 5y' + 6y = 0$ as a systems of differential equations with $y_1(t) = y(t), y_2(t) = y'(t)$ and solve with $y(0) = 2, y'(0) = -5$.

47. Suppose that $a_{n+1} = 3a_n - 4a_{n-1}$. Let $\vec{v}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Find a matrix A such that $\vec{v}_{n+1} = A\vec{v}_n$.